



TITLE:

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CITATION:

Funaki, Tadahisa. Derivation of variational problems from microscopic interface model (Viscosity Solutions of Differential Equations and Related Topics). 数理解析研究所講究録 2003, 1323: 76-83

ISSUE DATE:

2003-05

URL:

<http://hdl.handle.net/2433/43132>

RIGHT:

Derivation of variational problems from microscopic interface model

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1 Introduction

This note reviews recent results on the $\nabla\varphi$ interface model considered over the wall or under the weak effects of additional self-potentials. We are, in particular, interested in the scaling limit which passes from microscopic to macroscopic levels. The results are classified into two types:

1. Static results.
 - Large Deviation Principle
 - Derivation of Variational Problems (VP)
 - Wulff Shape, Winterbottom Shape
 - Alt-Caffarelli or Alt-Caffarelli-Friedman's VP
2. Dynamic results.
 - Motion by Mean Curvature (MMC) with anisotropy
 - Dynamic Large Deviation Principle
 - MMC with reflection
 - Evolutionary Variational Inequality
 - Fluctuation
 - Stochastic PDE with reflection

2 Static results

2.1. Let us introduce the **$\nabla\varphi$ interface model** briefly. We are concerned with a surface (interface) in \mathbb{R}^{d+1} , which separates two distinct pure phases, described by the height variables $\phi = \{\phi(x) \in \mathbb{R}, x \in \Gamma\}$ measured from a reference hyperplane Γ located in \mathbb{R}^d . To avoid complications, we assume that the interface has no overhangs nor bubbles. The variables ϕ are microscopic objects, and the space Γ is discretized and taken as $\Gamma = D_N (\equiv ND \cap \mathbb{Z}^d)$, lattice torus $(\mathbb{Z}/N\mathbb{Z})^d (\equiv \{1, 2, \dots, N\}^d)$ or \mathbb{Z}^d . Here D is a (macroscopic) bounded domain in \mathbb{R}^d and N represents the size of the microscopic system.

An **energy** is associated with each height variable $\phi : \Gamma \rightarrow \mathbb{R}$ as the sum over all bonds $\langle x, y \rangle$ in Γ (or in $\Gamma \cup \partial\Gamma$)

$$H(\phi) \equiv H_\Gamma^\psi(\phi) = \sum_{\langle x, y \rangle \subset \Gamma \text{ (or } \Gamma \cup \partial\Gamma)} V(\phi(x) - \phi(y)),$$

and the **equilibrium state (Gibbs measure)** is defined by

$$d\mu \equiv d\mu_\Gamma^\psi = Z^{-1} e^{-H(\phi)} \prod_{x \in \Gamma} d\phi(x),$$

where Z is a normalization constant. The **potential** V is symmetric, smooth and strictly convex ($0 <^{\exists} c_- \leq V'' \leq^{\exists} c_+ < \infty$). Note that the boundary conditions $\psi = \{\psi(x), x \in \partial\Gamma\}$ are required to define $H(\phi)$ and μ when $\Gamma = D_N$. An infinite volume limit (thermodynamic limit) as $\Gamma \nearrow \mathbb{Z}^d$ exists when $d \geq 3$ and the limit measure μ has a long correlation. More information on the $\nabla\varphi$ interface model can be found in [7] and [13].

2.2. Our main interest is in the **scaling limit**, which passes from microscopic to macroscopic levels, defined by

$$h^N(\theta) = \frac{1}{N} \phi([N\theta]), \quad \theta \in D \text{ (or } \in \mathbb{T}^d \equiv [0, 1]^d, \mathbb{R}^d),$$

where $[N\theta]$ stands for the integral part of $N\theta (\in \mathbb{R}^d)$ taken componentwise. The function h^N is the macroscopic height variable associated with the microscopic $\phi : \Gamma \rightarrow \mathbb{R}$. The **surface tension** $\sigma = \sigma(u)$ is the macroscopic energy for a surface with tilt $u \in \mathbb{R}^d$ determined by

$$\mu(\text{tilt of } h^N \sim u) \underset{N \rightarrow \infty}{\sim} \exp\{-N^d \sigma(u)\}.$$

Theorem 1. (Large Deviation, Deuschel-Giacomin-Ioffe [4]) Consider the Gibbs measure $\mu_{D_N}^0$ on $\Gamma = D_N$ with 0-boundary conditions $\psi(x) = 0, x \in \partial D_N$. Then, the probability that h^N is close to a given macroscopic surface $h \in H_0^1(D)$ behaves as

$$\mu_{D_N}^0(h^N \sim h) \underset{N \rightarrow \infty}{\sim} \exp\{-N^d \Sigma_D(h)\},$$

where $\Sigma_D(h)$ is the total surface tension of h defined by

$$(1) \quad \Sigma_D(h) = \int_D \sigma(\nabla h(\theta)) d\theta. \quad \square$$

This result is an analogue of Dobrushin-Kotecký-Shlosman [5] for the Ising model.

Corollary 2. (Wall and constant volume conditions) For every $v \geq 0$, under the conditional probability $\mu_{D_N}^0(\cdot | h^N \geq 0, \int_D h^N(\theta) d\theta \geq v)$, the law of large numbers $h^N \rightarrow \bar{h}_v$ (as $N \rightarrow \infty$) holds, where \bar{h}_v is the minimizer (called **Wulff shape**) of the variational problem

$$\min \left\{ \Sigma_D(h); h \in H_0^1(D), h \geq 0, \int_D h(\theta) d\theta = v \right\}. \quad \square$$

We add a **weak self-potential** term to the energy $H_{D_N}^\psi(\phi)$:

$$H_{D_N}^{\psi, Q, W}(\phi) = H_{D_N}^\psi(\phi) + \sum_{x \in D_N} Q\left(\frac{x}{N}\right) W(\phi(x)),$$

having the boundary conditions $\psi(x) = Nf(x/N), x \in \partial D_N$ determined from macroscopic function $f : \partial D \rightarrow \mathbb{R}$, where $Q : D \rightarrow [0, \infty)$ and $W : \mathbb{R} \rightarrow \mathbb{R}$ satisfies $\alpha_\pm = \lim_{r \rightarrow \pm\infty} W(r)$ such that $\alpha_+ \wedge \alpha_- \leq W \leq \alpha_+ \vee \alpha_-$. The Gibbs measure is associated and defined by

$$d\mu_{D_N}^{\psi, Q, W} = Z_{\psi, Q, W}^{-1} e^{-H_{D_N}^{\psi, Q, W}(\phi)} \prod_{x \in D_N} d\phi(x).$$

Theorem 3. (Large Deviation, Funaki-Sakagawa [11]) Assume $A := \alpha_+ - \alpha_- \geq 0$. Then,

$$\mu_{D_N}^{\psi, Q, W}(h^N \sim h) \underset{N \rightarrow \infty}{\sim} \exp\{-N^d I_D^A(h)\},$$

where

$$I_D^A(h) = \Sigma_D^A(h) - \inf_{h' \in H_f^1(D)} \Sigma_D^A(h'),$$

$$\Sigma_D^A(h) = \Sigma_D(h) - A \int_D Q(\theta) 1_{\{h(\theta) \leq 0\}} d\theta,$$

and $H_f^1(D)$ is the space of all $h \in H^1(D)$ having boundary conditions f . \square

Corollary 4. The law of large numbers $h^N \longrightarrow \bar{h}_A$ (as $N \rightarrow \infty$) holds under $\mu_{D_N}^{\psi, Q, W}$, where \bar{h}_A is the minimizer of the variational problem

$$\min \{ \Sigma_D^A(h); h \in H_f^1(D) \}. \quad \square$$

Remark 1. (1) The variational problem obtained in Corollary 4 was studied by Alt-Caffarelli [1], Alt-Caffarelli-Friedman [2], Weiss [18] and others.

(2) The large deviation for the Gibbs measure with δ -pinning instead of weak self-potentials is discussed by [11] in one dimension.

(3) Bolthausen-Ioffe [3] proved the law of large numbers for the Gibbs measure on the wall with δ -pinning and quadratic potential under the constant volume condition in two dimension. The limit called **Winterbottom shape** is uniquely (except translation) characterized by a certain variational problem. \square

3 Dynamic results

3.1. One can introduce **microscopic dynamics** (stationary and reversible under the Gibbs measure μ) for the interfaces by the SDEs (Langevin equation)

$$d\phi_t(x) = -\frac{\partial H}{\partial \phi(x)}(\phi_t) dt + \sqrt{2}dw_t(x), \quad x \in \Gamma,$$

where $\{w_t(x), x \in \Gamma\}$ is a family of independent Brownian motions and

$$\frac{\partial H}{\partial \phi(x)}(\phi) = \sum_{y: |x-y|=1} V'(\phi(x) - \phi(y)).$$

The goal is to discuss the **space-time diffusive scaling limit** for $\phi_t = \{\phi_t(x), x \in \Gamma\}$:

$$h^N(t, \theta) = \frac{1}{N} \phi_{N^2 t}([N\theta]).$$

Theorem 5. (Hydrodynamic Limit, Funaki-Spohn [12] on the torus \mathbb{T}^d , Nishikawa [14] on D with boundary conditions) As $N \rightarrow \infty$, $h^N(t, \theta) \longrightarrow h(t, \theta)$. The limit $h(t, \theta)$ is a unique weak solution of the nonlinear PDE (MMC with anisotropy):

$$\begin{aligned} (2) \quad \frac{\partial h}{\partial t}(t) &= \operatorname{div} \{ \nabla \sigma(\nabla h(t)) \} \\ &\equiv \sum_{i=1}^d \frac{\partial}{\partial \theta_i} \left\{ \frac{\partial \sigma}{\partial u_i}(\nabla h(t)) \right\}. \end{aligned} \quad \square$$

The surface tension has the following properties: $\sigma \in C^1(\mathbb{R}^d)$, $\nabla\sigma$ is Lipschitz continuous and σ is strictly convex. The PDE (2) can be regarded as the gradient flow for $\Sigma = \Sigma_{\mathbb{T}^d}$ or Σ_D , the total surface tension (1) on \mathbb{T}^d or D :

$$\frac{\partial h}{\partial t}(t) = -\frac{\delta}{\delta h(\theta)}\Sigma(h(t)).$$

Theorem 6. (Dynamic Large Deviation, Funaki-Nishikawa [9] on \mathbb{T}^d)

$$P(h^N(t) \sim h(t), t \leq T) \underset{N \rightarrow \infty}{\sim} \exp\{-N^d I_T(h)\},$$

where $h(t) = h(t, \theta)$ is a given motion of surface and

$$I_T(h) = \frac{1}{4} \int_0^T dt \int_{\mathbb{T}^d} \left[\frac{\partial h}{\partial t} - \operatorname{div} \{ \nabla \sigma(\nabla h(t)) \} \right]^2 d\theta.$$

The relation to the static large deviation (Theorem 1) is given by

$$\lim_{T \rightarrow \infty} S_T(\bar{h}) = \Sigma_{\mathbb{T}^d}(\bar{h}), \quad \bar{h} = \bar{h}(\theta),$$

where

$$S_T(\bar{h}) = \inf \{ I_T(h); h(T, \theta) = \bar{h}(\theta) \}. \quad \square$$

3.2. Dynamics on the wall are introduced by SDEs of Skorohod type:

$$(3) \quad d\phi_t(x) = -\frac{\partial H}{\partial \phi(x)}(\phi_t) dt + \sqrt{2} dw_t(x) + \frac{1}{N} f\left(\frac{t}{N^2}, \frac{x}{N}, \frac{1}{N} \phi_t(x)\right) dt + d\ell_t(x),$$

subject to the conditions

$$\phi_t(x) \geq 0, \quad \ell_t(x) \nearrow \quad \text{and} \quad \int_0^\infty \phi_t(x) d\ell_t(x) = 0,$$

where $f = f(t, \theta, h)$ is a given macroscopic external field. Note that $\ell_t(x)$ increases only when $\phi_t(x) = 0$. The unique invariant (stationary) measure (when $f = 0, \Gamma = D_N$ with 0-boundary conditions) is given by $\mu_{D_N}^0(\cdot | \phi \geq 0)$, which is reversible under the dynamics.

Theorem 7. (Hydrodynamic Limit, Funaki [8] on \mathbb{T}^d) As $N \rightarrow \infty$, $h^N(t, \theta) \rightarrow h(t, \theta)$. The limit $h(t, \theta)$ is a unique solution of the **evolutionary variational in-**

equality (MMC with reflection (obstacle)):

- (a) $h \in L^2(0, T; V), \frac{\partial h}{\partial t} \in L^2(0, T; V'), \quad \forall T > 0,$
- (b) $\left(\frac{\partial h}{\partial t}(t), h(t) - v \right) + (\nabla \sigma(\nabla h(t)), \nabla h(t) - \nabla v) \leq (f(t, h(t)), h(t) - v),$
a.e. $t, \quad \forall v \in V : v \geq 0,$
- (c) $h(t, \theta) \geq 0, \quad \text{a.e.},$
- (d) $h(0, \theta) = h_0(\theta),$

where $V = H^1(\mathbb{T}^d), H = L^2(\mathbb{T}^d), V' = H^{-1}(\mathbb{T}^d)$ and (\cdot, \cdot) denotes the inner product of H (or H^d) or the duality between V' and V . \square

Remark 2. Rezakhanlou [15], [16] derived a Hamilton-Jacobi equation under hyperbolic scaling from growing SOS dynamics ($\phi(x) \in \mathbb{Z}$) with constraints on the gradients (e.g. $\nabla \phi(x) \leq v$). Related results were obtained by Evans-Rezakhanlou [6] and Seppäläinen [17]. \square

Let us consider the **equilibrium dynamics** ϕ_t on the wall in one dimension, i.e., ϕ_t is a solution of the SDE (3) with $d = 1, \Gamma = \{1, 2, \dots, N-1\}, f = 0$ and with 0-boundary conditions $\phi_t(0) = \phi_t(N) = 0$ and an initial distribution $\mu_\Gamma^0(\cdot | \phi \geq 0)$. **Macroscopic fluctuation field** (around the hydrodynamic limit $h(t, \theta) = 0$) is defined by

$$\Phi^N(t, \theta) = \sqrt{N} h^N(t, \theta) (\geq 0), \quad \theta \in [0, 1].$$

Theorem 8. (Equilibrium Fluctuation, Funaki-Olla [10]) As $N \rightarrow \infty, \Phi^N(t, \theta) \Rightarrow \Phi(t, \theta)$. The limit $\Phi(t, \theta)$ is a unique weak stationary solution of the stochastic PDE with reflection of Nualart-Pardoux type:

$$\begin{aligned} \frac{\partial \Phi}{\partial t}(t, \theta) &= q \frac{\partial^2 \Phi}{\partial \theta^2}(t, \theta) + \sqrt{2} \dot{B}(t, \theta) + \xi(t, \theta), \quad \theta \in [0, 1], \\ \Phi(t, \theta) &\geq 0, \quad \int_0^\infty \int_0^1 \Phi(t, \theta) \xi(dtd\theta) = 0, \\ \Phi(t, 0) &= \Phi(t, 1) = 0, \quad \xi: \text{random measure}, \end{aligned}$$

where $\dot{B}(t, \theta)$ is a space-time white noise and

$$q = 1/\langle \eta^2 \rangle_\nu, \quad \nu(d\eta) = e^{-V(\eta)} d\eta \Big/ \int_{\mathbb{R}} e^{-V(\eta')} d\eta'. \quad \square$$

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